Simple Approximations for the Longitudinal Magnetic Polarizabilities of Some Small Apertures

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Abstract—Simple approximations are given for the longitudinal magnetic polarizabilities of some small apertures of various shapes, as functions of the aperture width to length ratios. The shapes considered are the rectangle, diamond, rounded end slot, and ellipse, of which only the last has an exact solution.

I. Introduction

In two recent papers [1], [2], polynomial approximations were given for the electric and transverse magnetic polarizabilities of small apertures of four different shapes. For completeness, it would be desirable to have similar expressions for the longitudinal magnetic polarizabilities. The aperture shapes being considered in this paper and the direction of the applied magnetic field for the longitudinal magnetic polarizabilities are shown in Fig. 1. In all cases $W \leqslant L$.

In [1] and [2], an important feature was that useful results were obtained simply from consideration of some of the properties which exact solutions should possess. A similar method for longitudinal magnetic polarizabilities has not yet been found; therefore in the formulation of approximate expressions more reliance has to be placed on numerical values. The most comprehensive set of calculated values seems to be that of De Smedt [3], [4], who has given the "dimensionless polarizability," defined as the aperture polarizability divided by $(\text{area})^{3/2}$, for a range of aspect ratios for all four shapes. In the case of the ellipse only, an exact solution is available [5], [6], and experimental results for the rectangle, rounded end slot, and some other shapes have been given by Cohn [7].

II. Form of Approximating Function

For each aperture shape, the longitudinal magnetic polarizability can be expressed in the form

$$P_m = f\left(\frac{W}{L}\right)L^3$$

in which $f(W/L)$ is a dimensionless polarizability coefficient. In the following, the aspect ratio W/L is designated α, and the range of α is from 0 to 1.

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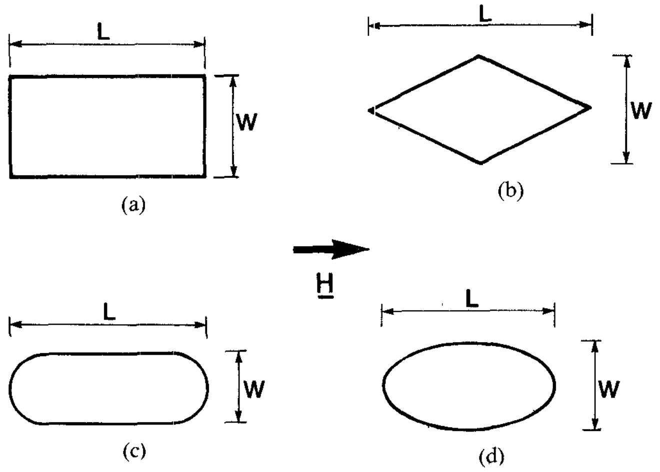

Fig. 1. Aperture shapes and direction of applied magnetic field. (a) Rectangle. (b) Diamond. (c) Rounded end slot. (d) Ellipse.

Typical curves of $f(\alpha)$ versus α show smooth, gradually increasing functions for α values in the range of 0.1 to 1.0 [7]. For all aperture shapes $f(\alpha)$ must go to zero as α goes to zero, for in that case the aperture closes up and the polarizability goes to zero.

The first derivative $f'(\alpha)$ of the polarizability coefficient for each shape being considered can be expected to go to infinity as α goes to zero. The reasoning which leads to this conclusion is as follows. Consider first the rectangle, and enclose it between the largest possible inscribed ellipse, which will have the same L and W as for the rectangle, and an exterior ellipse which passes through all four corners of the rectangle. It can be shown that if the outer ellipse has the same aspect ratio and orientation as the inner ellipse, its major and minor axes will be $\sqrt{2}\,L$ and $\sqrt{2}\,W$, respectively. The longitudinal magnetic polarizability of the rectangle will lie between those of the interior ellipse and the exterior ellipse. (Consider the electrolytic tank analog experiment of Cohn [7]. If a metallic obstacle representing a rectangular aperture had material removed to make it into the inscribed ellipse, then the conductance would have decreased; conversely, if material had been added to make it larger, then the conductance would have increased.) Because small aperture polarizabilities vary as the cube of a characteristic dimension for apertures of identical shapes, and the polarizability of the inner ellipse goes to

$$-\frac{\pi L^3}{24 \ln \alpha} \quad \text{as} \quad \alpha \to 0$$

(see Section IV below) then the polarizability of the rectangle is bounded by

$$-\frac{2\sqrt{2}\,\pi L^3}{24 \ln \alpha} \text{ and } -\frac{\pi L^3}{24 \ln \alpha} \text{ as } \alpha \to 0.$$

While those bounds are too far apart to give useful numerical values, they do indicate that the first derivative of the polarizability coefficient with respect to α goes to infinity as α goes to zero. Similar reasoning applies to the diamond and the rounded end slot.

The range of α of interest in this paper is from 0 to 1, i.e., for $W \leqslant L$. However in principle the ranges of α for the rectangle, diamond, and ellipse are from 0 to ∞, with the section from 1 to ∞ corresponding to the transverse magnetic polarizability, whereas for the rounded end slot $\alpha > 1$ has no meaning. The quadratic behavior of the transverse magnetic polarizability coefficient for small α [2] corresponds to a linear behavior for large α when interpreted as a longitudinal polarizability coefficient. (The coefficients are multiplied by the cubes of different dimensions to obtain the aperture polarizabilities.) Accordingly the longitudinal polarizability coefficient for the rectangle, diamond, and ellipse should go to a linear function as α goes to infinity.

A simple function which has the three desired properties:

$$f(\alpha) \to 0 \quad \text{as} \quad \alpha \to 0$$

$$f'(\alpha) \to \infty \quad \text{as} \quad \alpha \to 0$$

and

$$f(\alpha) \to \text{constant} \times \alpha \quad \text{as} \quad \alpha \to \infty$$

is

$$f(\alpha) = \frac{a}{\ln\left(1 + \frac{b}{\alpha}\right)}$$

in which a and b are constants and ln denotes the natural logarithm. The linear dependence for large α is a consequence of

$$\ln(1+x) \to x \quad \text{as} \quad x \to 0.$$

A simple routine was used to obtain the constants a and b for each aperture shape by fitting to De Smedt's numerical values obtained from [3]. Note that if instead the approximating function was a polynomial or a rational function comprising one polynomial divided by another, then as α goes to zero the approximating function would go to α^n, where n is a positive integer, and the ratio of (exact value)/(approximate value) would go to infinity.

III. Results

For the rectangle, De Smedt [3] gives numerical values for the dimensionless polarizabilities for α equal to 0.1, 0.2, 0.333, 0.5, 0.75, 0.8, and 1.0. The corresponding polarizability coefficients were calculated, and the approximation function

$$f(\alpha) = \frac{0.132}{\ln\left(1 + \frac{0.660}{\alpha}\right)}$$

gives agreement of better than 0.4 percent for those α values. It also gives agreement of better than 0.9 percent with Cohn's experimental results for $\alpha = 0.1$, 0.15, 0.2, 0.3, 0.5, 0.75, and 1.0 [7].

For the diamond De Smedt's calculations are for the same α values as the rectangle, and the corresponding approximation function is

$$f(\alpha) = \frac{0.109}{\ln\left(1 + \frac{2.33}{\alpha}\right)}$$

with a discrepancy of less than 1.3 percent.

In the case of the rounded end slot, the computed values from [3] are for $\alpha = 0.1$, 0.2, 0.333, 0.5, and 0.8 and the exact value is known for $\alpha = 1.0$, which is a circle. The approximation

$$f(\alpha) = \frac{0.195}{\ln\left(1 + \frac{2.12}{\alpha}\right)}$$

provides agreement of better than 3.1 percent with De Smedt's calculated values and 3.8 percent with Cohn's experimental results for α equal to 0.1, 0.15, 0.2, 0.3, 0.5, 0.75, and 1.0. These are the largest discrepancies for any of the four shapes and are probably caused by the large α behavior of the approximating function not being applicable to the rounded end slot. The agreement for this shape can be improved if a small quadratic term is included in the numerator. The approximation function

$$f(\alpha) = \frac{0.187 + 0.052\alpha(1-\alpha)}{\ln\left(1 + \frac{2.12}{\alpha}\right)}$$

gives agreement of better than 1.4 percent with De Smedt's calculated values and 1.7 percent with Cohn's experiments.

For the ellipse, which has an exact solution (see Section IV below) an approximation function was derived using the exact values for $\alpha = 0.1$, 0.2, 0.3, 0.5, 0.8, and 1.0. Although much more information is available for the ellipse, those α values were selected to be typical in number and spacing of those used for the other shapes, so that the ellipse could be used as a test case. The approximation function derived from those values was

$$f(\alpha)=\frac{0.115}{\ln\left(1+\frac{1.00}{\alpha}\right)}$$

with an error of less than 0.9 percent.

This function was then checked against the exact values at $\alpha=0.4$, 0.6, 0.7, and 0.9, and the maximum error was still less than 0.9 percent. Such good agreement is to be expected on that smooth part of the curve. The next check was for the very small α region, between 0 and 0.1. The error was 1.8 percent at $\alpha=0.06$, and 4.1 percent at $\alpha=0.02$. In the limit as $\alpha\to 0$, the error goes to 12 percent.

For this particular case of the ellipse, it is possible to improve the accuracy for extremely small α by setting the numerator of the approximating function to 0.131 ($=\pi/24$) and adjusting the other parameter. However the improved accuracy for very small α is at the expense of decreased accuracy for larger α; therefore a better approach for small α is to use the known approximation given in Section IV below.

It is likely that similar percentage errors for extremely small α values would also occur for the other three shapes. Such α values are not likely to be found in microwave devices, but could arise in electromagnetic compatibility calculations for the electromagnetic field penetration through a crack or imperfect seam. Fortunately the accuracy required in those applications is usually not as stringent as in microwave devices. Another consideration is that of tolerances and measurement accuracy. The fact that aperture polarizabilities vary as the cube of a characteristic dimension is recognized as leading to significant tolerancing problems in some microwave devices. The errors referred to above for extremely small values of α are associated with $f'(\alpha)$ approaching ∞, the most extreme tolerance and measurement situation possible.

IV. Polarizability of an Ellipse

The exact longitudinal magnetic polarizability coefficient for an ellipse, in the notation of this paper, is

$$\frac{\pi(1-\alpha^2)}{24\left[K\left(\sqrt{1-\alpha^2}\right)-E\left(\sqrt{1-\alpha^2}\right)\right]}$$

which for small α is approximately equal to

$$\frac{\pi(1-\alpha^2)}{24[\ln(4/\alpha)-1]}.$$

The error for this approximation at $\alpha=0.3$ is 3.7 percent; at $\alpha=0.2$ it is 1.5 percent and at $\alpha=0.1$ it is 0.33 percent. At extremely small values of α the $\ln\alpha$ term dominates in the denominator and the function goes to

$$-\frac{\pi}{24\ln\alpha}.$$

The exact transverse magnetic polarizability coefficient for an ellipse was given in [2] and the electric polarizability coefficient which was referred to in [1] but not stated is

$$\frac{\pi\alpha^2}{24E\left(\sqrt{1-\alpha^2}\right)}.$$

In all of these expressions K and E are the complete elliptic integrals of the first and second kinds, respectively, as defined in [8]. It should be noted that the definitions in [8] are not the same as in [9]. This difference is known to workers in the field but may not be more widely known.

V. Conclusions

Simple approximations have been given for the longitudinal magnetic polarizability coefficients of some small apertures. Multiplication of the coefficients by L^3 gives the polarizabilities of the respective apertures. Although some reasoning was used to select a suitable form of approximation function, that form is not unique; nor does there seem to be an associated physical interpretation (such as the polarizability per unit length used in [1] and [2]).

The expressions have been obtained by a curve-fitting process applied to De Smedt's numerical data. Accordingly the absolute accuracy is a combination of the accuracies of the curve fitting and the data. The latter are understood to have an accuracy of the order of 1 percent.

Although the large α behavior was used to select a suitable form of approximation function for three of the shapes, no data points were used for $\alpha>1$ and accordingly better accuracy is obtained by regarding cases of $\alpha>1$ as transverse polarizabilities and using the expressions in [2].

VI. Postscript

Since the above material was prepared, the author has become aware of another two sets of calculated values for the polarizability coefficients of rectangular apertures [10], [11]. In particular the electric and magnetic coefficients for a square are given in [11] as 0.1138 and 0.2600, respectively, compared with 0.1126 used in [1] and 0.2596 used in [2] and in this paper.

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